Exercise 6A: Solutions

1 a As m and n are even, $m = 2p$ and $n = 2q$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} m + n &= 2p + 2q \\ &= 2(p + q), \end{aligned}$$

is an even number.

b As m and n are even, $m = 2p$ and $n = 2q$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} mn &= (2p)(2q) \\ &= 4pq \\ &= 2(2pq), \end{aligned}$$

is an even number.

2 As m and n are odd, $m = 2p + 1$ and $n = 2q + 1$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} m + n &= (2p + 1) + (2q + 1) \\ &= 2p + 2q + 2 \\ &= 2(p + q + 1), \end{aligned}$$

is an even number.

3 As m is even and n is odd, $m = 2p$ and $n = 2q + 1$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} mn &= 2p(2q + 1) \\ &= 2(2pq + p), \end{aligned}$$

is an even number.

4 a If m is divisible by 3 and n is divisible by 7, then $m = 3p$ and $n = 7q$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} mn &= (3p)(7q) \\ &= 21pq, \end{aligned}$$

is divisible by 21.

b If m is divisible by 3 and n is divisible by 7, then $m = 3p$ and $n = 7q$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} m^2 n &= (3p)^2(7q) \\ &= 9p^2(7q) \\ &= 63p^2 q \end{aligned}$$

is divisible by 63.

5 If m and n are perfect squares then $m = a^2$ and $n = b^2$ for some $a, b \in \mathbb{Z}$. Therefore,

$$mn = (a^2)(b^2) = (ab)^2,$$

is also a perfect square.

6 Expanding both brackets gives,

$$\begin{aligned} (m + n)^2 - (m - n)^2 &= m^2 + 2mn + n^2 - (m^2 - 2mn + n^2) \\ &= m^2 + 2mn + n^2 - m^2 + 2mn - n^2 \\ &= 4mn, \end{aligned}$$

which is divisible by 4.

7 (Method 1) If n is even then n^2 is even and $6n$ is even. Therefore the expression is of the form

$$\text{even} - \text{even} + \text{odd} = \text{odd}.$$

(Method 2) If n is even then $n = 2k$ where $k \in \mathbb{Z}$. Then

$$\begin{aligned} n^2 - 6n + 5 &= (2k)^2 - 6(2k) + 5 \\ &= 4k^2 - 12k + 5 \\ &= 4k^2 - 12k + 4 + 1 \\ &= 2(2k^2 - 6k + 2) + 1, \qquad = \end{aligned}$$

is odd.

8 (Method 1) If n is odd then n^2 is odd and $8n$ is even. Therefore the expression is of the form

$$\text{odd} + \text{even} + \text{odd} = \text{even}.$$

(Method 2) If n is odd then $n = 2k + 1$ where $k \in \mathbb{Z}$. Then

$$\begin{aligned} n^2 + 8n + 5 &= (2k+1)^2 + 8(2k+1) + 3 \\ &= 4k^2 + 4k + 1 + 16k + 8 + 3 \\ &= 4k^2 + 20k + 12 \\ &= 2(2k^2 + 10k + 6), \end{aligned}$$

is even.

9 First suppose n is even. Then $5n^2$ and $3n$ are both even. Therefore the expression is of the form

$$\text{even} + \text{even} + \text{odd} = \text{odd}.$$

Now suppose n is odd. Then $5n^2$ and $3n$ are both odd. Therefore the expression is of the form

$$\text{odd} + \text{odd} + \text{odd} = \text{odd}.$$

10 Firstly, if $x > y$ then $x - y > 0$. Secondly, since x and y are positive, $x + y > 0$. Therefore,

$$\begin{aligned} x^4 - y^4 &= (x^2 - y^2)(x^2 + y^2) \\ &= (x-y)(x+y)(x^2+y^2) \\ &= \overbrace{(x-y)}^{\text{positive}}\overbrace{(x+y)}^{\text{positive}}\overbrace{(x^2+y^2)}^{\text{positive}} \\ &> 0. \end{aligned}$$

Therefore, $x^4 > y^4$.

11 We have,

$$\begin{aligned} x^2 + y^2 - 2xy &= x^2 - 2xy + y^2 \\ &= (x-y)^2 \\ &\geq 2xy. \end{aligned}$$

Therefore, $x^2 + y^2 \geq 2xy$.

12a We prove that Alice is a knave, and Bob is a knight.

Suppose Alice is a knight
$\Rightarrow$ Alice is telling the truth
$\Rightarrow$ Alice and Bob are both knaves
$\Rightarrow$ Alice is a knight and a knave
This is impossible.
$\Rightarrow$ Alice is a knave
$\Rightarrow$ Alice is not telling the truth
$\Rightarrow$ Alice and Bob are not both knaves
$\Rightarrow$ Bob is a knight
$\Rightarrow$ Alice is a knave, and Bob is a knight

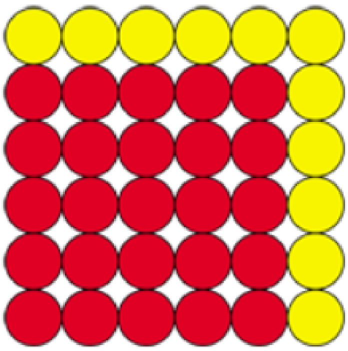
b We prove that Alice is a knave, and Bob is a knight.

Suspose Alice is a knight
$\Rightarrow$ Alice is telling the truth
$\Rightarrow$ They are both of the same kind
$\Rightarrow$ Bob is a knight
$\Rightarrow$ Bob is lying
$\Rightarrow$ Bob is a knave
$\Rightarrow$ Bob is a knight and a knave.
This is impossible.
$\Rightarrow$ Alice is a knave
$\Rightarrow$ Alice is not telling the truth
$\Rightarrow$ Alice and Bob are of a different kind
$\Rightarrow$ Bob is a knight
$\Rightarrow$ Alice is a knave, and Bob is a knight

c We will prove that Alice is a knight, and Bob is a knave.

Suppose Alice is a knave
$\Rightarrow$ Alice is not telling the truth
$\Rightarrow$ Bob is a knight
$\Rightarrow$ Bob is telling the truth
$\Rightarrow$ Neither of them are knaves
$\Rightarrow$ Both of them are knights
$\Rightarrow$ Alice is a knight and a knave
This is impossible.
$\Rightarrow$ Alice is a knight
$\Rightarrow$ Alice is telling the truth
$\Rightarrow$ Bob is a knave
$\Rightarrow$ Bob is lying
$\Rightarrow$ At least one of them is a knave
$\Rightarrow$ Bob is a knave
$\Rightarrow$ Alice is a knight, and Bob is a knave.

13a In the diagram below, there are 11 yellow tiles. We can also count the yellow tiles by subtracting the number of red tiles, 5^2, from the total number of tiles, 6^2. Therefore $11 = 6^2 - 5^2$.



b Every odd number is of the form $2k+1$ for some $k \in \mathbb{Z}$. Moreover,

$$\begin{aligned}(k+1)^2 - k^2 &= k^2 + 2k + 1 - k^2 \\ &= 2k+1,\end{aligned}$$

so that every odd number can be written as the difference of two squares.

c Since $101 = 2 \times 50 + 1$, we have,

$51^2 - 50^2 = 101.$

14a Since

$$\frac{9}{10} = \frac{99}{110} \text{ and } \frac{10}{11} = \frac{100}{110},$$

it is clear that

$$\frac{10}{11} > \frac{9}{10}.$$

b We have,

$$\begin{aligned}\frac{n}{n+1} - \frac{n-1}{n} &= \frac{n^2}{n(n+1)} - \frac{n(n-1)}{n(n+1)} \\ &= \frac{n^2 - n(n-1)}{n(n+1)} \\ &= \frac{n^2 - n^2 + n}{n(n+1)} \\ &= \frac{1}{n(n+1)} \\ &> 0\end{aligned}$$

since $n(n+1) > 0$. Therefore,

$$\frac{n}{n+1} > \frac{n-1}{n}.$$

15a We have,

$$\begin{aligned}\frac{1}{10} - \frac{1}{11} &= \frac{11}{110} - \frac{10}{110} \\ &= \frac{1}{110} \\ &< \frac{1}{100},\end{aligned}$$

since $110 > 100$.

b We have,

$$\begin{aligned}\frac{1}{n} - \frac{1}{n+1} &= \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)} \\ &= \frac{n+1-n}{n(n+1)} \\ &= \frac{1}{n(n+1)}, \\ &= \frac{1}{n^2+n}, \\ &< \frac{1}{n^2},\end{aligned}$$

since $n^2 + n > n^2$.

16 We have,

$$\begin{aligned}\frac{a^2+b^2}{2} - \left(\frac{a+b}{2}\right)^2 &= \frac{a^2+b^2}{2} - \frac{(a+b)^2}{4} \\ &= \frac{2a^2+2b^2}{4} - \frac{a^2+2ab+b^2}{4} \\ &= \frac{2a^2+2b^2-a^2-2ab-b^2}{4} \\ &= \frac{a^2-2ab+b^2}{4} \\ &= \frac{(a-b)^2}{4} \\ &\geq 0.\end{aligned}$$

17a Expanding gives,

$$\begin{aligned}(x-y)(x^2+xy+y^2) &= x^3+x^2y+xy^2-x^2y-xy^2-y^3\\ &= x^3-y^3,\end{aligned}$$

which is the difference of two cubes.

b Completing the square by treating y as a constant gives,

$$\begin{aligned}x^2+yx+y^2 &= x^2+yx+\frac{y^2}{4}-\frac{y^2}{4}+y^2\\ &= \left(x^2+yx+\frac{y^2}{4}\right)+\frac{3y^2}{4}\\ &= \left(x+\frac{y}{2}\right)^2+\frac{3y^2}{4}\\ &\geq 0\end{aligned}$$

c Firstly, if $x \geq y$ then $x-y \geq 0$. Therefore,

$$\begin{aligned}x^3-y^3 &= \overbrace{(x-y)}^{\geq 0}\overbrace{(x^2+xy+y^2)}^{\geq 0}\\ &\geq 0.\end{aligned}$$

Therefore, $x^3 > y^3$.

18a Let D be the distance to and from work. The time taken to get to work is $D/12$ and the time taken to get home from work is $D/24$. The total distance is $2D$ and the total time is

$$\begin{aligned}\frac{D}{12}+\frac{D}{24} &= \frac{2D}{24}+\frac{D}{24}\\ &= \frac{3D}{24}\\ &= \frac{D}{8}\end{aligned}$$

The average speed will then be

$$\begin{aligned}\text{distance} \div \text{time} &= 2D \div \frac{D}{8}\\ &= 2D \times \frac{8}{D}\\ &= 16 \text{ km/hour}.\end{aligned}$$

b Let D be the distance to and from work. The time taken to get to work is D/a and the time taken to get home from work is D/b. The total distance is $2D$ and the total time is

$$\begin{aligned}\frac{D}{a}+\frac{D}{b} &= \frac{bD}{ab}+\frac{aD}{ab}\\ &= \frac{aD+bD}{ab}\\ &= \frac{(a+b)D}{ab}\end{aligned}$$

The average speed will then be

$$\begin{aligned}\text{distance} \div \text{time} &= 2D \div \frac{(a+b)D}{ab}\\ &= 2D \times \frac{ab}{(a+b)D}\\ &= \frac{2ab}{a+b} \text{ km/hour}.\end{aligned}$$

c We first note that $a+b>0$. Secondly,

$$\begin{aligned}\frac{a+b}{2}-\frac{2ab}{a+b} &= \frac{(a+b)^2}{2(a+b)}-\frac{4ab}{2(a+b)}\\ &= \frac{(a+b)^2-4ab}{2(a+b)}\\ &= \frac{a^2+2ab+b^2-4ab}{2(a+b)}\\ &= \frac{a^2-2ab+b^2}{2(a+b)}\\ &= \frac{(a-b)^2}{2(a+b)}\\ &\geq 0\end{aligned}$$

since $(a-b)\geq 0$ and $a+b>0$. Therefore,

$$\frac{a+b}{2}\geq\frac{2ab}{a+b}.$$